Scaling of Impulse Responses and Pulse Responses Calculated via Discrete Fourier Transforms

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1.0 Motivation

This document derives the scaling to be applied to an impulse response or pulse calculated continuous frequency domain values using an inverse discrete Fourier transform (DFT) such as an inverse fast Fourier transform (FFT). While at first it would seem that the obvious answer to this question would be the standard “divide by N”, this is not necessarily true because the input values for the inverse DFT were calculated from the continuous frequency domain, and not the direct result of a DFT. It is therefore desireable to to make sure of the result by deriving it from first principles.

2.0 Impulse Response

The basic definition of the continuous time Fourier transform and its inverse are

$$H(f) = \mathcal{F}(h(t)) \equiv \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}dt \tag{EQ 1}$$

$$h(t) = \mathcal{F}^{-1}(H(f)) \equiv \int_{-\infty}^{\infty} H(f)e^{j2\pi ft}df \tag{EQ 2}$$

This particular form of the equations was taken largely from [1]; however, the signs in the exponents were changed to conform to the more generally accepted definition for the Fourier transform. Also, the following development also generally follows that in [1] except that is has been restated to bring out specific points.

If the continuous time and frequency are replaced by sampled time and frequency via the substitutions $t = n\Delta t$ and $f = m\Delta f$, then these integrals can be approximated by the sums

$$H(m\Delta f) \approx \sum_{n=-\infty}^{\infty} h(n\Delta t)e^{-j2\pi mn\Delta f\Delta t}\Delta t \tag{EQ 3}$$

$$h(n\Delta t) \approx \sum_{m=-\infty}^{\infty} H(m\Delta f)e^{j2\pi mn\Delta f\Delta t}\Delta f \tag{EQ 4}$$

Supposing further that Δt and Δf are related by the equation

$$\Delta f = \frac{1}{N\Delta t} \tag{EQ 5}$$

Where for convenience N is an even number, then these sums become

$$H(m\Delta f) \approx \sum_{n=-\infty}^{\infty} h(n\Delta t)e^{-j2\pi mn/N}\Delta t \tag{EQ 6}$$

$$h(n\Delta t) \approx \sum_{m=-\infty}^{\infty} H(m\Delta f)e^{j2\pi mn/N}\Delta f \tag{EQ 7}$$

If $h(n\Delta t)$ is zero outside the interval $0 \le t < N\Delta t$ and we define $h_n \equiv h(n\Delta t)$, then

$$H(m\Delta f) \approx \Delta t \sum_{n=0}^{N-1} h_n e^{-j2\pi mn/N} \tag{EQ 8}$$

This, then, motivates the definition of the DFT

$$H_m \equiv \sum_{n=0}^{N-1} h_n e^{-j2\pi mn/N} \tag{EQ 9}$$

and the confusing part is the relationship between the continuous frequency domain and the sampled data frequency domain, which then becomes

$$H(m\Delta f) \approx \Delta t H_m \tag{EQ 10}$$

The inverse DFT converts from the sampled data frequency domain to the sampled data time domain. It is

$$h_n = \frac{1}{N} \sum_{m=0}^{N-1} H_m e^{-j2\pi mn/N} \tag{EQ 11}$$

The conversion from the continuous frequency domain to the continuous time domain is therefore (approximately)

$$h(n\Delta t) \approx \frac{1}{N\Delta t} \sum_{m=0}^{N-1} H(m\Delta f)e^{-j2\pi mn/N} \tag{EQ 12}$$

Equation 12 is the desired equation for the impulse response, and differs from the inverse DFT by the $\frac{1}{\Delta t}$ conversion factor from the discrete frequency domain to the continuous frequency domain. If *N=samples*nbits* and Δt*=1/(rate*samples)* then the factor $1/N\Delta t$ becomes *rate/nbits*.

3.0 Pulse Response

The pulse response is the convolution of a rectangular waveform with the impulse response. The usual way to compute it is to transform into the frequency domain, multiply the spectral density of the pulse times the transfer function, and then transform back to the time domain.

Given a pulse *p(t)* with unit amplitude and duration *T*, the spectral density is

$$P(f) = \mathcal{F}(p(t)) = \int_{-\infty}^{\infty} p(t)e^{-j2\pi ft}dt \qquad \text{(EQ 13)}$$

Since the pulse has unit amplitude in the range $-\frac{T}{2} \le p(t) \le \frac{T}{2}$ and is zero elsewhere, equation 13 becomes

$$P(f) = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{-j2\pi ft}dt = \int_{0}^{\frac{T}{2}} 2\cos(2\pi ft)dt = \frac{2}{2\pi f}\sin(2\pi ft)\Big|_{0}^{\frac{T}{2}} = \frac{1}{\pi f}\sin(\pi fT) \qquad \text{(EQ 14)}$$

$$P(f) = T\frac{\sin(\pi fT)}{\pi fT} \qquad \text{(EQ 15)}$$

Equation 15 is the equation that is normally referred to as “sine x over x” or “synch”; however, for the purposes of this document, the normalizing factor of *T* in this equation is every bit as important.

For a channel with transfer function *H(f)*, the pulse response therefore becomes

$$p(t) = \mathcal{F}^{-1}\left(T\frac{\sin(\pi fT)}{\pi fT}H(f)\right) \qquad \text{(EQ 16)}$$

The calculation for the conversion from the continuous frequency domain to the continuous time domain is therefore

$$p(n\Delta t) \approx \frac{T}{N\Delta t}\sum_{m=0}^{N-1} \frac{\sin(\pi fT)}{\pi fT}H(m\Delta f)e^{-j2\pi mn/N} \qquad \text{(EQ 17)}$$

Similar to the calculation for the impulse response, if
*N=samples*nbits* , Δt*=1/(rate1*samples)*, and *T=1/rate2*,
then the factor $T/N\Delta t$ becomes *rate1/rate2/nbits*.

4.0 References

[1] Press, Teukolsky, Vetterling and Flannery, “Numerical Recipes in C++ - The Art of Scientific Computing”, Chapter 12, Second Edition, Cambridge University Press, copyright 2002.

