What’s a Smith Chart?

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1.0 Introduction

The primary purpose of this document is to describe how to display S parameters on a Smith chart. Rather than going immediately into the mechanics of the process, however, it is useful to understand where the Smith chart came from and how it is used, thus providing a motivation for the display itself. This document therefore starts with a brief historical background and a description of the underlying mathematics before describing the display itself.

2.0 Historical Background

The Smith chart [1] was developed to help design impedance matching networks. At the time (end of WWII), microwave circuits could only function over relatively narrow bandwidths, and the general approach was to impedance match at the center of the band, build a prototype, measure the bandwidth, and hope for the best. Reflection coefficient magnitude and phase were measured by probing the signal strength along a slotted transmission line or waveguide, plotting these measurements, and then extracting the magnitude and phase from the plot. This is where the term Voltage Standing Wave Ratio (VSWR, pronounced *viz-wahr*) came from. At the time, neither the circuit construction technologies nor the measurement technologies nor the understanding of microwave circuit design supported more than a narrow band approach.

Especially for a single design frequency, the Smith chart was extremely useful because it allowed the circuit designer to easily predict how a circuit element would affect the reflection coefficient. One could therefore very quickly identify the several circuit topologies that could be expected to produce the desired result, and choose which of those best suited the application. Once the topology was chosen, the most common approach was to build a prototype and tweak it to see what performance one could get. This is what one did to design amplifiers, mixers, or antenna drive networks - anything for which a good impedance match was required from a complex structure.

As a side note, microwave filter design is almost an entirely separate field in which the control of bandwidth is the essential requirement. In microwave filter design, however, the structures are simpler and more readily analyzed, and therefore there is a richer structure of analytical constructs available. It would take a lot of mental gymnastics to apply the Smith chart to any but the simplest filters, and therefore the Smith chart is not generally used in microwave filter design.

By the time I served my microwave apprenticeship in 1974-77, the most significant change that had occurred in the design of microwave impedance matching networks was the invention of the harmonic converter. The harmonic converter downconverts multiple signals at a single microwave frequency at a time down to a fixed intermediate frequency while maintaining the phase relationship between them. A harmonic converter is combined with microwave directional couplers on one side and a display device on the other to form a microwave network analyzer, or what is now called a Vector Network Analyzer (VNA). What makes the harmonic converter so useful is the fact that it can adapt to the frequency of measurement over a very broad range of frequencies without needing a separate sample of the measurement signal itself. Thus, it became very easy to perform swept measurements of the reflection or transmission coefficient and display them on a Smith chart.

At the time, computers were still run primarily in batch mode, however, so the Smith chart was still the method of choice for most microwave design tasks.

3.0 Construction and Application

A Smith chart is a polar plot of the reflection coefficient, overlaid on a set of impedance axes which have been distorted to show the reflection coefficient those impedances would produce.

That's all there is to it. No more, no less. The beauty of the Smith chart lies in its simplicity, and in particular in the elegance of the mathematics on which it's based.

3.1 Fundamentals

Given a complex source impedance Z_0 , usually positive real, and a complex load impedance Z, the reflection coefficient is [2]

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} \qquad \text{(EQ 1)}$$

As noted in [2], a reflection coefficient of zero is obtained when $Z = Z_0$; however, maximum power transfer occurs when $Z = Z_0^*$. By far the most common case is when Z_0 is real and positive, in which case maximum power transfer occurs when the reflection coefficient is zero. This is the case plotted in the Smith chart.

One could in principle generate the axes for a Smith chart by applying equation 1 directly to a rectangular grid in the impedance domain, and those axes would be correct. These grid lines, in the usual order of precedence are

1. $Z = jB$ $\quad -\infty < B < \infty$ This forms the outer boundary of the chart, corresponding to a reflection coefficient magnitude of 1 and all possible phases.
2. $Z = A$ $\quad 0 < A < \infty$ This forms a horizontal line through the center of the diagram.

3. $Z = R_0 + jB \qquad -\infty < B < \infty$ This forms a circle which goes from 0 to 1 and back to 0, centered on the horizontal axis.
4. $Z = A \pm jR_0 \qquad 0 < A < \infty$ This forms two arcs- one from *-j* to 1 and one from *+j* to 1.
   This completes what is usually considered to be the simplest form of Smith chart.
5. $Z = \frac{n}{N}R_0 + jB \qquad -\infty < B < \infty$ This forms some more circles centered on the horizontal axis and passing through 1.
6. $Z = A \pm j\frac{n}{N}R_0 \qquad 0 < A < \infty$ This forms more arcs passing through 1.
   A typical Smith chart will have many such circles or arcs to form a more or less uniform grid in the Smith chart itself.

Figure 1 illustrates this transformation for the case of $N = 2$.

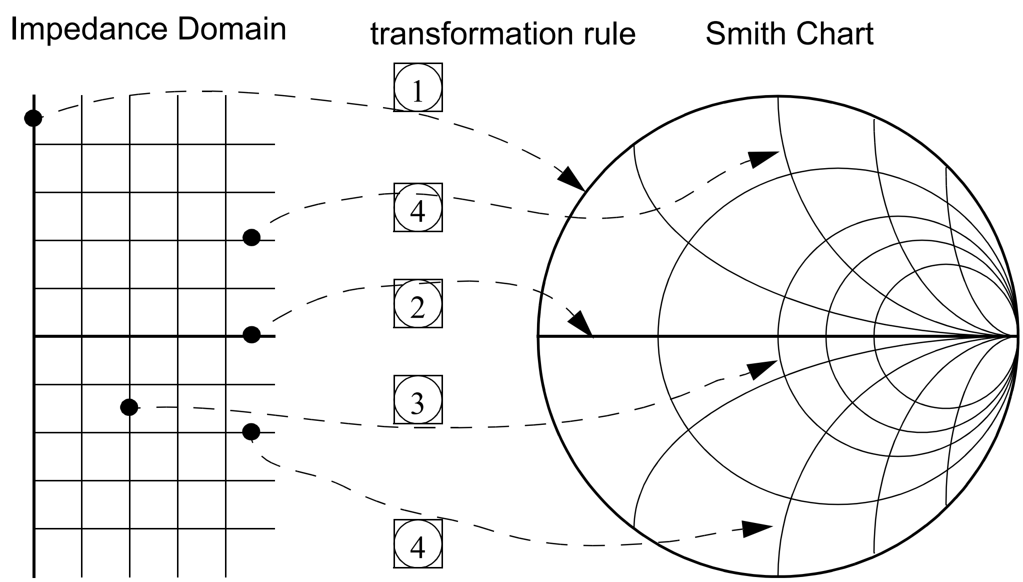

FIGURE 1. Transformation from Impedance Domain to Smith Chart

3.2 Application

With one more bit of background, it will be possible to illustrate a non-trivial application of the Smith chart. Note that there is an equation for reflection coefficient that is analogous to equation 1 except that it uses admittances instead of impedances:

$$\Gamma = \frac{Y_0 - Y}{Y_0 + Y} \qquad \text{(EQ 2)}$$

Application of this equation to the admittance domain results in a chart which looks like the Smith chart except that it's mirror-imaged. Therefore, whereas the Smith chart makes it easy to see what will happen to the serial combination of two impedances, its mirror image makes it easy to see what will happen to the parallel combination of two impedances.

Figure 2 illustrates a Smith chart application that was typical in my early days as a microwaver. Suppose one wanted to impedance match the input to a microwave transistor. The input impedance of the transistor typically has a large real part and an inconveniently small capacitive imaginary part. The transistor package, however, introduces a significant phase shift, resulting in a single-frequency reflection coefficient shown by the *X* in Figure 2. One way to impedance match the transistor at this frequency is to apply a shunt inductor followed by a series inductor. The left side of Figure 2 illustrates the mirror-imaged (admittance-based) Smith chart and the right side illustrates the Smith chart in its usual orientation.In the chart on the lefthand side, a shunt inductance is used to shift the reflection in a counterclockwise direction. In the chart on the righthand side, a series inductor is added to rotate the reflection coefficient in to the origin. In each case, the reflection coefficient follows one of the grid lines in the chart.

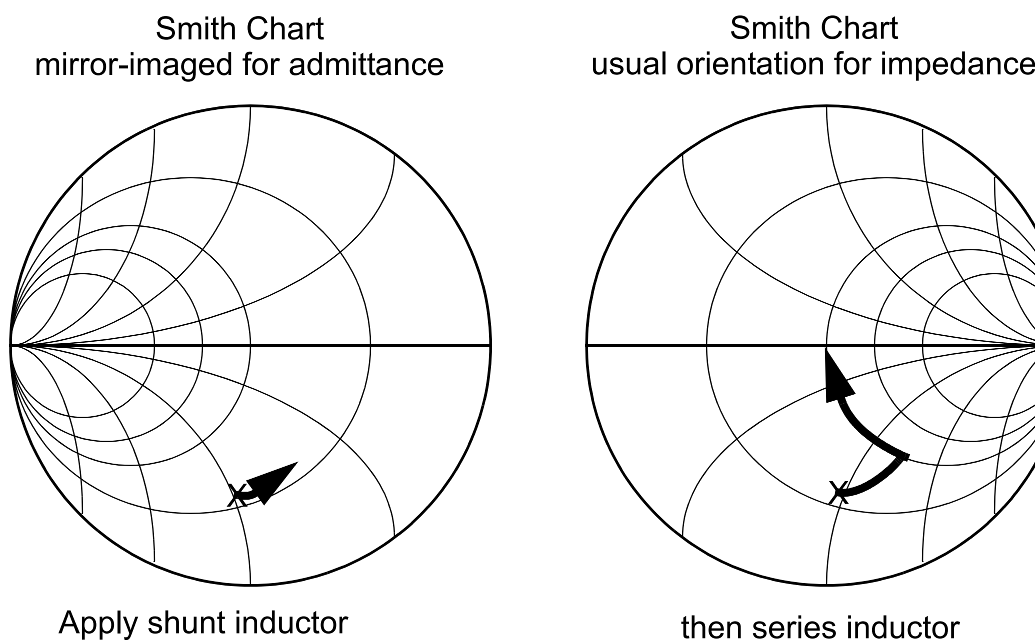

FIGURE 2. Typical Smith chart application: Impedance matching a microwave transistor

The resulting circuit is shown in Figure 3.

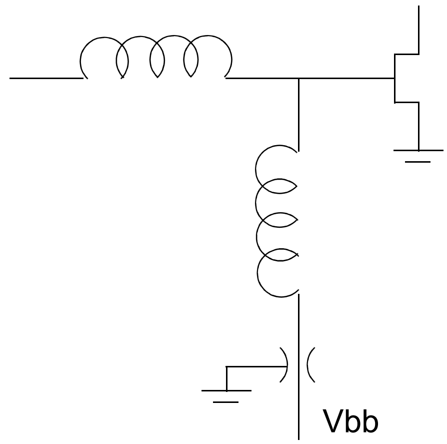

FIGURE 3. Microwave transistor with input impedance matching

One could have also impedance matched this transistor by using series inductance followed by shunt inductance. (Exercise left to the reader.) The choice of topology depends primarily on the transmission line impedances that are available and the mechanical construction that will be most practical.

3.3 The Elegant Part

At this point, one of the elegant aspects of the Smith chart should be readily apparent. Both equation 1 and equation 2 are bilinear transforms, and therefore have the property that they transform straight lines into circles and circles into straight lines. Thus, drawing the grid lines on a Smith chart is entirely a matter of drawing circles in the right place.

As shown in detail in Annex A, for $Z = A + jB$ and characteristic impedance R_0, the radii and centers of these circles are as shown in Table 1.

TABLE 1. Radii and centers of Smith chart circles

Description	Center	Radius
Constant real part	$\frac{A}{A + R_0}$	$\frac{R_0}{A + R_0}$
Constant imaginary part	$\frac{jB - R_0}{jB}$	$\frac{R_0}{B}$

4.0 Displaying S Parameters

One of the goals of the exposition above was to motivate two key points:

1. Display reflection coefficients such as S_{11} and S_{22} by plotting the value at each frequency as a polar plot on a Smith chart because that will help the user understand the impedance matching implications of the data.
2. Display transmission coefficients such as S_{21} and S_{12} by plotting the value at each frequency as a polar plot on a polar scale. This will help the user understand the transmission delay and attenuation vs. frequency. The user will probably also want to see the magnitude of transmission coefficient plotted as dB vs. frequency. By its very nature, the Smith chart is not applicable to transmission coefficients.

I have a tablet of Smith charts that are approximately 24" in diameter and, with a copyright of 1949, older than most of the employees at SiSoft. It's also about as detailed as a Smith chart should ever get. It uses the following scale

TABLE 2. Smith chart scale, normalized to R_0

Range	Step
0-0.2	0.01
0.2-0.5	0.02
0.5-1.0	0.05
1-2	0.1
2-5	0.2
5-10	1.0
10-20	2.0
20-50	10

An on-screen display should probably have about a tenth of the subdivisions for the full chart, evolving to the scale described in Table 2 above for an expanded view.

5.0 References

[1] Smith, P. H.; An Improved Transmission Line Calculator; Electronics, Vol. 17, No. 1, pg. 130, January 1944.

[2] Matthei, G. L., Young, L., and Jones, E. M. T.; *Microwave Filters, Impedance-matching Networks, and Coupling Structures*; Section 2.08, pg. 34, McGraw-Hill, 1964.

6.0 Annex A: Derivation of Bilinear Equations

For constant real part and imaginary part between minus infinity and infinity

$$\Gamma = \frac{A + jB - R_0}{A + jB + R_0} \qquad \text{(EQ 3)}$$

For infinite imaginary part, the resulting value is 1. For zero imaginary part, the value is $\frac{A-R_0}{A+R_0}$. Since the center is half way in between, the center is at

$$\frac{1+\frac{A-R_0}{A+R_0}}{2} = \frac{A}{A+R_0} \qquad \text{(EQ 4)}$$

and the radius is at

$$1-\frac{A}{A+R_0} = \frac{R_0}{A+R_0} \qquad \text{(EQ 5)}$$

The vector between all other points in the contour and the center of the contour is

$$\frac{A+jB-R_0}{A+jB+R_0}-\frac{A}{A+R_0} = \frac{A^2+jBA+jBR_0-R_0^2-A^2-jBA-AR_0}{(A+R_0)\cdot(A+jB+R_0)} \qquad \text{(EQ 6)}$$

$$= \frac{-R_0}{A+R_0}\cdot\frac{A-jB+R_0}{A+jB+R_0} \qquad \text{(EQ 7)}$$

thus demonstrating that the contour is in fact a circle with radius and center as calculated.

The solution for constant imaginary part and variable real part is a little different. Suppose that the center were at $\frac{jB-R_0}{jB}$. Then with respect to that center, the contour would be

$$\frac{A+jB-R_0}{A+jB+R_0}-\frac{jB-R_0}{jB} = \frac{jAB-B^2-jBR_0-jAB+AR_0+B^2+jBR_0-jBR_0+R_0^2}{(jB)\cdot(A+jB+R_0)} \qquad \text{(EQ 8)}$$

$$= \frac{R_0}{jB}\cdot\frac{A-jB+R_0}{A+jB+R_0} \qquad \text{(EQ 9)}$$

Thus, the contour is an arc of a circle, the hypothesized center is in fact the center of the circle, and the radius is $\frac{R_0}{B}$ $B \neq 0$. For $B = 0$, the contour is a straight line.